Passivity:

- Consider

$$(*)\quad \begin{aligned} \dot{x} &= f(x,u) \qquad & x \in \mathbb{R}^n \\ y &= h(x,u) \qquad & y,u \in \mathbb{R}^m \end{aligned}$$

where $f(0,0) = 0$, $h(0,0) = 0$

- $(*)$ is passive if $\exists$ a positive semidefinite function $V(x)$ ($V(x) \geq 0$, $V(0) = 0$) s.t.

storage function

$$\underbrace{\dot{V}}_{\frac{\partial V}{\partial x} f(x,u)} \leq u^T y$$

- Output strictly passive:

$$\dot{V} \leq u^T y - y^T \varphi(y) \quad \text{where} \quad y^T \varphi(y) > 0 \quad \forall\, y \neq 0$$

- strictly passive

$$\dot{V} \leq u^T y - W(x) \quad \text{where} \quad W(x) > 0 \quad \forall\, x \neq 0$$

Properties of passive systems:

Lemma 6.6:

passive $\Longrightarrow$ stable with $u=0$

proof:

$$\dot{V} \leq u^T y \overset{u=0}{=} 0 \Longrightarrow \text{stable}$$

- To have A.S. we need stronger conditions.
- If the sys. is strictly passive, then

$$\dot{V} \leq u^T y - W(x) = -W(x) < 0 \quad \forall x \neq 0$$

- But we also need to make sure that V is positive-definite, i.e. $V(x) > 0 \ \forall x \neq 0$
- Strict passivity is only possible if V is p.d.
- proof by contradiction

- Suppose $\exists\ \bar{x} \neq 0$ s.t. $V(\bar{x}) = 0$

- Start the sys at $x(0) = \bar{x}$. Then

$$\dot{V}(x(t)) \leq -W(x(t))$$

$$\Rightarrow V(x(t)) - \underbrace{V(x(0))}_{V(\bar{x})=0} \leq -\int_0^t W(x(s))\,ds$$

$$\Rightarrow \int_0^t W(x(s))\,ds \leq -V(x(t)) \leq 0$$

- Because $W(x) > 0\ \forall x \neq 0$, this is only possible if $W(x(t)) = 0 \Leftrightarrow x(t) = 0\ \forall t$

$$\Rightarrow \bar{x} = 0 \quad \text{※}$$

Contradicts $\bar{x} \neq 0$

- Therefore, $V(x)$ is p.d. ✓

<u>Lemma 6.7 (a)</u>

strict passive $\Rightarrow$ A.S.

- Can we have A.S. if the sys is output strictly passive?

- We need additional observability condition.

<u>Def</u>: (*) is <u>zero-state observable</u> if when $u=0$

$$y(t)=0 \quad \forall t \implies x(t)=0 \quad \forall t$$

<u>Lemma 6.7 (b)</u>

output strictly passive + zero-state observable $\implies$ A.S.

<u>proof</u>:

$$\dot{V} \leq -y^T \varphi(y) \quad \text{where } y^T\varphi(y) > 0 \quad \forall y \neq 0$$

- We use LaSalle's invariance principle to show A.S.

- $x(t) \to$ largest invariant set in $E = \{x \mid \dot{V}(x) = 0\}$

$$\dot{V}(x(t)) = 0 \implies y(t)^T \varphi(y(t)) = 0$$

$$\implies y(t) = 0$$

Zero-state obs.

$$\implies x(t) = 0 \quad \checkmark$$

- We also need to show V is p.d., which follows from the same argument as before for strict passivity.

<u>Example</u>:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a x_1^3 - k x_2 + u$$

$$y = x_2$$

- $a, k > 0$

- Take $V(x) = \frac{1}{2} x_2^2 + \frac{1}{4} a x_1^4$

$$\implies \dot{V} = -k x_2^2 + u x_2 = -k y^2 + u y$$

$\Longrightarrow$ output strictly passive.

- To check zero-state obs.

$$y(t) = 0 \Longrightarrow x_2(t) = 0$$

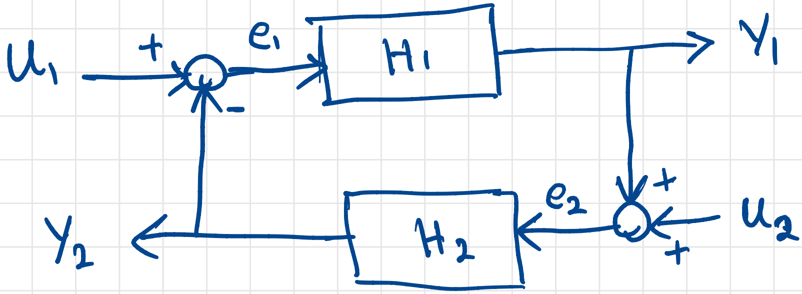
$$\Longrightarrow a x_1^3(t) = 0$$

$$\Longrightarrow x_1(t) = 0 \quad \checkmark$$

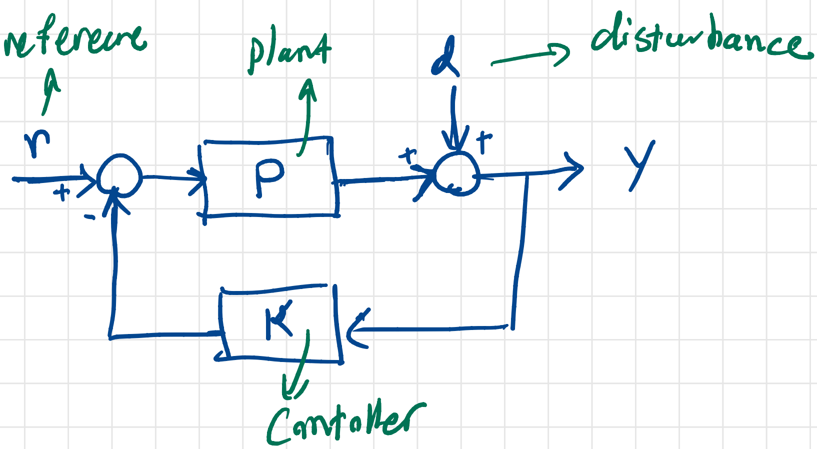
- Therefore, it is AS.
- Actually, it is GAS because V is radially unbounded.

Passivity theorems:

- Analyze stability of Feedback sys using passivity of each sub-system



- Example



- P is complicated, but passive
- Design K based on some approximation of P

Passivity thm: if K is Passive, then the FB system is Passive

Thm 6.1:

- The feedback connection of two passive systems, is passive.

Proof:

- H_1 and H_2 are passive. Therefore $\exists V_1, V_2$ s.t.

$$\dot{V}_1 \leq e_1^T y_1$$
$$\dot{V}_2 \leq e_2^T y_2$$

- Take $V = V_1 + V_2$. Then,

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \leq e_1^T y_1 + e_2^T y_2$$
$$= (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2$$
$$= u_1^T y_1 + u_2^T y_2$$
$$= \underbrace{[u_1, u_2]}_{u} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} \Rightarrow \text{passive}$$

Stability of FB sys. :

- If H_1 and H_2 are passive $\Rightarrow$ FB is passive $\xRightarrow{\text{Lem. 6.6}}$ stable when $u=0$
- To have A.S. we need the two sub sys. to be either strictly passive or output strictly passive and zero-state obs.

Thm 6.3 :

For FB sys. with $u=0$, $x=0$ is A.S if

a) both sys are strictly passive

or

b) both sys are output S.P. and zero-state obs.

or

c) one sys is strictly passive and the other one is output S.P. and zero state obs proof for (5):

$$\dot{V}_1 \leq e_1^T Y_1 - W_1(x_1)$$
$$\dot{V}_2 \leq e_2^T Y_2 - Y_2^T \varphi(Y_2)$$

- Take $V = V_1 + V_2$

$$\dot{V} \leq e_1^T Y_1 + e_2^T Y_2 - W_1(x_1) - Y_2^T \varphi(Y_2)$$
$$= u^T Y - W_1(x_1) - Y_2^T \varphi(Y_2)$$
$$= -W_1(x_1) - Y_2^T \varphi(Y_2)$$

- $\dot{V} = 0 \implies \left.\begin{array}{l} x_1 = 0 \\ Y_2 = 0 \implies x_2 = 0 \end{array}\right\} \implies$ A.S.

L_2-Stability of FB sys

- We showed stability of $x=0$ when $u=0$
- What if u are disturbances and we want to show y is small if disturbances are small. ⤳ L-stability

Thm 6.2:

- Assume H_1 and H_2 satisfy

$$\dot{V}_1 \le e_1^T y_1 - \varepsilon_1 \|e_1\|^2 - \delta_1 \|y_1\|^2$$
$$\dot{V}_2 \le e_2^T y_2 - \varepsilon_2 \|e_2\|^2 - \delta_2 \|y_2\|^2$$

for some storage functions V_1, V_2

- Then, the FB sys is L_2-stable with finite gain if

$$\varepsilon_1 + \delta_2 > 0 \,, \quad \varepsilon_2 + \delta_1 > 0$$

Note: The constants ε_i, δ_i can be negative

special case: $\varepsilon_1 = \varepsilon_2 = 0$, $\delta_1, \delta_2 > 0$ $\Rightarrow$ H_1, H_2 are output s.p.

proof of the special case : (Lemma 6.8)

Take $V = V_1 + V_2$

$$\Rightarrow \dot{V} \leq u^T y - \delta_1 \|y_1\|^2 - \delta_2 \|y_2\|^2$$

$$\leq u^T y - \delta \|y\|^2$$

where $\delta = \min(\delta_1, \delta_2) > 0$

- using $u^T y \leq \frac{1}{2\alpha}\|u\|^2 + \frac{\alpha}{2}\|y\|^2$

$$\dot{V} \leq \frac{1}{2\alpha}\|u\|^2 - \left(\delta - \frac{\alpha}{2}\right)\|y\|^2$$

- Integrating with time

$$\left(\delta - \frac{\alpha}{2}\right)\int_0^\infty \|y\|^2 dt \leq \frac{1}{2\alpha}\int_0^\infty \|u\|^2 dt + V(x_0)$$

- Take $\alpha = \delta$, then

$$\|y\|_{L_2}^2 \leq \frac{1}{\delta^2}\|u\|^2 + \frac{2}{\delta}V(x_0)$$

$$\Rightarrow \|y\|_{L_2} \leq \frac{1}{\delta}\|u\| + \sqrt{\frac{2}{\delta}V(x_0)} \rightarrow \text{finite gain } L_2\text{-stable}$$